The magnetic response of disordered metallic rings: the large contribution of the far levels

M. Schechter[1], Y. Oreg[2], Y. Imry[2], and Y. Levinson[2]
[1] *The Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel*
[2] *Department of Condensed Matter Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*

We calculate the orbital linear magnetic response of disordered metallic rings to an Aharonov-Bohm flux using the BCS model for attractive electron-electron interaction. The contribution of all levels including those up to a high energy cutoff results in a much larger value than previously obtained using the local interaction model. The possible relevance of our results to the resolution of the discrepancy between the experimental and theoretical values for the ensemble-averaged persistent currents in these systems is discussed.

One of the remarkable phenomena of mesoscopic physics is the existence of equilibrium persistent currents in small normal metal rings, in the regime where the elastic mean free path l is much smaller than the ring's circumference L. This was both predicted theoretically [1] and observed experimentally [2–7]. While the phenomenon and the periodicity in the flux are well understood by now, the magnitude of the ensemble-averaged persistent current found experimentally is much larger than that obtained theoretically using the model of non-interacting electrons [8,9]. An attempt [10,11] to account for this discrepancy by the inclusion of electron-electron ("e-e") interactions [12] indeed increased the theoretical value, but still came short by a factor of about 5 [13].

Along with the large value of the ensemble-averaged persistent current observed, its sign poses significant questions as well. While in the first experiment [2] the sign of the ensemble-averaged persistent current was tentatively identified as diamagnetic, it was not determined definitively (see Ref. [13]). Later experiments reported a predominantly diamagnetic ensemble-averaged persistent current [5–7]. Theoretically, the paramagnetic/diamagnetic sign follows for repulsive/attractive effective e-e interactions (the interplay between the renormalized [14] repulsive e-e interaction and the phonon mediated attractive interaction determines the sign of the effective interaction [13]). Thus, the diamagnetic sign of the persistent currents observed in Gold [6], Copper [2], and Silver [7] suggests that these materials are very weak superconductors [15].

The model used to treat both signs of the interaction was that of local e-e interaction. The discrepancy between the theoretical values and the experimental results exists for repulsive [10], as well as attractive interactions [13]. For the repulsive case, the higher order renormalization which reduces the first order result is responsible for this discrepancy. For the attractive interaction case, the higher order renormalization does not reduce, and even somewhat increases the first order result. However, the restriction on the bare interaction constant, due to the experimental fact that Gold, Copper and Silver are not superconducting in observed temperatures, limits the value of the theoretical result in this case. For the magnetic response (the derivative of the persistent current at zero flux), there exist a further logarithm in the numerator [10], irrespective of the above renormalization of the effective interaction. Here we show that the use of the BCS Hamiltonian to model attractive e-e interaction gives rise to a much larger cutoff of this logarithm, and hence to a significantly larger value for the magnetic response to flux. The enhancement of the magnetic response is a result of the contribution of high energy levels (to be called "far levels") originating from pairing correlations which persist up to the energy of the Debye frequency $\omega_{\rm D}$. Thus, correlations on an energy scale much larger than the Thouless energy affect the magnetic response.

The contribution of the far levels was investigated in connection to superconductivity in small grains [16]. There, the correlations of levels much further than the superconducting gap from the Fermi energy $E_{\rm F}$, up to $\omega_{\rm D}$ prove to be significant. Using an exact solution of the reduced BCS Hamiltonian [17,18], it was shown that this contribution results in a much larger condensation energy than that given by the BCS value, in a wide parameter regime in which superconducting correlations are well developed. The far levels also affect single particle properties of a superconducting grain [19], such as the Matveev-Larkin parameter [20].

The difference between the local interaction model and the BCS Hamiltonian lies in the q dependence of the interaction in each, where q is the sum of the incoming (outgoing) momenta of the pair scattering. While the local interaction model assumes that the bare interaction does not depend on any momenta, and specifically on q, the BCS Hamiltonian assumes a sharp cutoff at small total momentum, and considers the interaction to be a delta function at $q = 0$. While for physical properties related to the superconducting phase of the system, near and below $T_{\rm c}$ (such as $T_{\rm c}$ itself) the two models give similar results, we show that for the magnetic response in the perturbative regime (equivalent to $T \gg T_{\rm c}$) the results in the two models differ substantially. We discuss the relation between the two models, and the effect of

relaxing the BCS assumption by taking the interaction to have finite width in q.

Recently it was suggested that the largeness of the observed current is due to AC noise [21,22] or interactions of the electrons with two level systems [23], and a relation to dephasing was suggested (one should also keep in mind that the latter interactions may lead to an additional attractive interaction). We, however, consider $T = 0$, and do not include dephasing (we consider the dephasing length L_ϕ to be larger than all relevant lengths). A finite L_ϕ will result in a suppression of the persistent current, exponential in L/L_ϕ.

We consider a quasi one dimensional disordered ring penetrated by a constant Aharonov-Bohm (AB) flux Φ in its middle. We calculate the magnetic response, denoted by F_2, to first order in the e-e interaction, for attractive interaction using the reduced BCS Hamiltonian.

The result we obtain with logarithmic accuracy is

$$\langle F_2 \rangle \equiv \left\langle \left. \frac{dI}{d\Phi} \right|_{\Phi=0} \right\rangle = \frac{8\pi\lambda E_{\rm Th}}{\Phi_0^2} \ln \frac{E_{\rm co}}{d} \ , \qquad (1)$$

where λ is the dimensionless pairing parameter [see Eq. (6)], d is the level spacing, $E_{\rm Th} = l^2/(s\tau L^2)$ is the Thouless energy, which is the inverse of the time to circulate the ring ($s = 1, 2, 3$ is the effective dimension of the ring for diffusive motion), $\Phi_0 = hc/(2e)$ and $\langle \cdots \rangle$ denotes ensemble averaging. The upper logarithmic cutoff $E_{\rm co} \equiv \min[\omega_{\rm D}, 1/\tau]$ represents the fact that energies (measured from the $E_{\rm F}$) further than $E_{\rm Th}$, up to $E_{\rm co}$ (far levels) contribute to the magnetic response (τ is the elastic mean free time). Their contribution enhances F_2 by about an order of magnitude due to the much larger logarithmic cutoff as compared to $E_{\rm Th}$ of the known result [10] [for $1/(\tau E_{\rm Th}) \approx 10^4$ the ratio of the logarithms is roughly 10].

In order to obtain Eq. (1) we consider the general Hamiltonian

$$H = \sum_\alpha \int dr\, \psi_\alpha^\dagger(r) \left[\frac{1}{2m} (\vec{P} - \frac{e}{c}\vec{A})^2 + U(r) \right] \psi_\alpha(r) + H_{\rm ee} \, , \qquad (2)$$

where $U(r)$ is the external potential which includes the disorder, and $H_{\rm ee}$ represents the e-e interaction. In the London gauge $\vec{A} = \Phi/(2\pi\rho)\hat{\phi}$ where ρ is the distance from the origin and $\hat{\phi}$ is in the clockwise direction of the ring. The free energy of the system and the persistent current are flux dependent, and related by $I(\Phi) = -dF/d\Phi$. By time reversal symmetry $I(0) = 0$, and for small flux

$$F(\Phi) = F_0 - \frac{1}{2} F_2 \Phi^2 + ... \, ; \qquad \left. \frac{dI}{d\Phi} \right|_{\Phi=0} = F_2 \, . \qquad (3)$$

We now turn to the calculation of F_2 to first order in the interaction. Perturbative analysis of the reduced BCS attractive interaction [Eq. (6) below] is valid, at $T = 0$, for $\lambda < 1/\ln(\omega_{\rm D}/d)$ [16] and at finite temperature ($T > d$) for $T \gg T_{\rm c}$. We take as the unperturbed Hamiltonian

$$H_0 = \sum_\alpha \int dr\, \psi_\alpha^\dagger(r) \left[\frac{P^2}{2m} + U(r) \right] \psi_\alpha(r) \qquad (4)$$

and the magnetic field together with the e-e interaction as perturbation

$$H_I = H_{\rm ee} + \qquad (5)$$
$$\sum_\alpha \int dr\, \psi_\alpha^\dagger(r) \left[-\frac{e}{2mc} (\vec{P}\cdot\vec{A} + \vec{A}\cdot\vec{P}) + \frac{e^2}{2mc^2} A^2 \right] \psi_\alpha(r) \, .$$

We calculate perturbatively the energy to third order in $H_{\rm I}$, and consider only the terms which are second order in the flux and first order in the interaction.

We denote by $|i\rangle$ the eigenstates of the noninteracting electrons in the disordered ring without magnetic field. In this basis, within the reduced BCS model, we obtain

$$H_{\rm I}^{\rm BCS} = -\sum_{ij\alpha} \frac{e\Phi}{mcL} P_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \lambda d {\sum_{ij}}' c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger c_{j\downarrow} c_{j\uparrow} \, . \quad (6)$$

Here c_i destroys an electron in the state $|i\rangle$ with wavefunction $\chi_i(r)$ and $P_{ij} = \langle i|P_\parallel|j\rangle$ is the matrix element of the momentum parallel to the ring's direction. The second sum is restricted to levels within $\omega_{\rm D}$ of $E_{\rm F}$. Note that contrary to the usual convention, λ is negative for attractive interaction. We assume that the width of the ring is much smaller than its radius [24]. The A^2 term is then interaction independent. We choose the χ_i's to be real, and then P_{ij} is pure imaginary and $P_{ii} = 0$.

Using third order perturbation theory [25] we find that for $T = 0$ to first order in λ

$$F_2 = 8 \left(\frac{e}{mcL} \right)^2 \lambda d \sum_{\substack{0<i<\omega_{\rm D} \\ -\omega_{\rm D}<j<0}} \frac{|P_{ij}|^2}{\omega_{ij}^2} \, , \qquad (7)$$

where $0 < i < \omega_{\rm D}$ represents states whose energy is between $E_{\rm F}$ and $E_{\rm F} + \omega_{\rm D}$, and $\omega_{ij} = \epsilon_i - \epsilon_j$. Performing disorder averaging over $|P_{ij}|^2$ we obtain

$$\left\langle |P_{ij}|^2 \right\rangle = \frac{p_{\rm F}^2 d\tau}{\pi(1 + \omega_{ij}^2\tau^2)s} \, , \qquad (8)$$

which is roughly constant for $\omega_{ij} < 1/\tau$ and zero for $\omega_{ij} > 1/\tau$. In Eq. (7), for energies smaller than $\omega_{\rm D}$ and $1/\tau$ there is a double sum and a second power of energy in the denominator. The result is therefore a logarithm with an upper cutoff which is given by $\min[\omega_{\rm D}, 1/\tau]$. We evaluate the sum in Eq. (7) taking for simplicity the noninteracting spectrum to be equally spaced, and obtain Eq. (1).

This equation differs from the result of Ambegaokar and Eckern (AE) [10] by the upper cutoff of the logarithm, being $\min[\omega_{\rm D}, 1/\tau]$ in comparison to $E_{\rm Th}$ in

Ref. [10]. We now explain the origin of this difference in some detail. AE consider the same Hamiltonian as in our Eq. (2), only with local e-e interaction $V(r_1 - r_2) = \tilde{\lambda} N(0)^{-1} \delta(r_1 - r_2)$. In terms of the noninteracting eigenstates $H_{\rm ee}$ is then given by

$$H_{\rm ee}^{\rm local} = \frac{1}{2} \sum_{ijkl,\sigma} V_{ijkl} c_{i\sigma}^{\dagger} c_{j\bar{\sigma}}^{\dagger} c_{k\bar{\sigma}} c_{l\sigma} \ , \tag{9}$$

where $V_{ijkl} = \tilde{\lambda} N(0)^{-1} \int \chi_i(r)\chi_j(r)\chi_k(r)\chi_l(r) dr$ and $\sigma \neq \bar{\sigma}$ due to the interaction being local. The dependence of the energy on flux, to first order in the interaction, can be written as

$$\Delta F = \sum_{i,j<0} V_{ijji}(\Phi) = \sum_{i \neq j<0} V_{ijji}(\Phi) + \sum_{i<0} V_{iiii}(\Phi) \ . \tag{10}$$

We now show that the second, diagonal part contribution to the magnetic response corresponds to the BCS result as given in Eq. (1). ΔF can be expanded to second order in the flux to obtain F_2 for the local interaction model by following the same procedure done above for the reduced BCS Hamiltonian, with $H_{\rm ee}$ given by Eq. (9). The result is

$$\langle F_2 \rangle = 4 \left(\frac{e}{mcL} \right)^2 \left\langle \sum_{j<0, i>0} \frac{|P_{ij}|^2}{\omega_{ij}^2} \sum_{l<0} (V_{jllj} - V_{illi}) \right\rangle \ . \tag{11}$$

The diagonal contribution to this result is given by the $l = i$ terms, and since to leading order $V_{iiii} - V_{ijji} = 2\tilde{\lambda} d$ [26] we obtain for the diagonal contribution in the local interaction model the same expression as Eq. (7), without the explicit cutoff at $\omega_{\rm D}$. Due to Eq. (8), the diagonal contribution gives Eq. (1) with $E_{\rm co} = 1/\tau$. Therefore, the BCS approximation is equivalent to assuming that the flux dependence of the offdiagonal matrix elements is small, and can be neglected. In the case of the local interaction model this is not the case, and the contribution of the offdiagonal elements to the magnetic response is significant, and opposite in sign to that of the diagonal element. This results in a partial cancellation, and therefore a reduction of the high logarithmic cutoff in Eq. (1).

Another way to understand the relation between the two models is to consider the q dependence of the interaction in both (where q is the sum of the incoming momenta). AE derive their result as a sum over all q's (see Eqs. [(12)-(14)] of Ref. [10]), and the result within the BCS Hamiltonian [Eq. (1)] corresponds to their $q = 0$ term. For the local interaction model, the contribution to the magnetic response of the high energies, above $E_{\rm Th}$ exactly cancels between equal magnitude and opposite signs of the $q = 0$ term and the sum of all $q \neq 0$ terms. *This makes the q independent assumption for the bare interaction crucial.* The existence of excess interaction at small total momentum q would thus significantly affect the result. The BCS interaction assumes just that, the existence of excess interaction at $q = 0$.

A physical justification for taking a q dependent interaction can be obtained, for the attractive phonon mediated interaction, from the usual restriction that all (incoming and outgoing) scattered states are within $\omega_{\rm D}$ of $E_{\rm F}$ (see e.g. Ref. [27]). This restriction implies a significant q dependence on the scale of $q_c = \omega_{\rm D}/v_{\rm F}$. One can therefore take a q dependent attractive interaction in the form of a step function with width q_c. For $q_c < 2\pi/L$ one finds that only the $q = 0$ term survives, and the BCS result for the magnetic response is recovered.

For metals such as Gold, Copper and Silver, this condition requires the circumference of the ring to be smaller than $0.2 - 0.4 \mu m$, which is an order of magnitude smaller than the relevant experimental lengths [2,5,7]. The effects of relaxing this condition, i.e. having $q_c \gtrsim 2\pi/L$, as well as the effect of the q dependence on the persistent current itself will be considered elsewhere [28].

The central result of this Letter is the large interaction correction to the derivative of the persistent current at zero flux, within the BCS model, as is given in Eq. (1). Physically, this is due to the large contribution of the far levels, up to $\min[\omega_{\rm D}, 1/\tau]$ from the Fermi level.

Our calculations were done for $T = 0$. At finite temperature $T < \min[\omega_{\rm D}, 1/\tau]$ (but neglecting dephasing, assuming $L_\phi \gg L$) the magnetic response would be given by Eq. (1) with d replaced by T. Therefore, on top of the large magnetic response, we predict a weak, logarithmic, temperature dependence of the magnetic response up to $T \approx \min[\omega_{\rm D}, 1/\tau]$. The weak temperature dependence is due to the addition of the many small contributions, all having the same sign, of the levels up to the large energy cutoff (see also Refs. [16,29]).

The local interaction model and the reduced BCS Hamiltonian are two models used to describe the effective e-e interaction resulting from the retarded electron-phonon interaction. Though different, these two models give similar results for many properties of the superconductor near and below $T_{\rm c}$. The robustness of the physics near and below the transition temperature makes the differences between the two models irrelevant. However, the magnetic response in the perturbative regime is a more subtle property, that distinguishes between the two models.

Our result for the reduced BCS Hamiltonian also follows by expanding Richardson's exact solution [17]. This has the analytical merit of being a first order expansion of an exact solution, as well as an additional viewpoint on the contribution of the far levels, and will be given elsewhere [30].

We would like to thank O. Agam and I. Aleiner for discussing with us the special role of the $q = 0$ term in the calculation of the persistent current within the local interaction model. We benefited from valuable discussions with N. Argaman, V. Kravtsov, D. Orgad, A. Punnoose, Y. Aharonov, B. Altshuler, R. Berkovits, H. Bouchiat, P.

Brouwer, J. von Delft, A. Finkel'stein, Y. Gefen, D. E. Khmelnitskii, M. Khodas, F. von Oppen, Z. Ovadyahu, R. Smith, U. Smilansky, R. A. Webb, and H. A. Weidenmueller. M.S. is thankful for the support by the Lady Davis fund. This work was supported by a Center of Excellence of the Israel Science Foundation (ISF), Jerusalem and by the German Federal Ministry of Education and Research (BMBF) within the Framework of the German-Israeli Project Cooperation (DIP), by the German-Israeli Foundation(GIF), and by ISF grant number 160-01/1.

[1] M. Buttiker, Y. Imry, and R. Landauer, Phys. Lett. A **96**, 365 (1983).
[2] L. P. Levy, G. Dolan, J. Dunsmuir, and H. Bouchiat, Phys. Rev. Lett. **64**, 2074 (1990).
[3] V. Chandrasekhar, R. A. Webb, M. J. Brady, M. B. Ketchen, W. J. Gallagher, and A. Kleinsasser, Phys. Rev. Lett. **67**, 3578 (1991).
[4] D. Mailly, C. Chapelier, and A. Benoit, Phys. Rev. Lett. **70**, 2020 (1993).
[5] B. Reulet, M. Ramin, H. Bouchiat, and D. Mailly, Phys. Rev. Lett. **75**, 124 (1995).
[6] E. M. Q. Jariwala, P. Mohanty, M. B. Ketchen, and R. A. Webb, Phys. Rev. Lett. **86**, 1594 (2001).
[7] R. Deblock, R. Bel, B. Reulet, H. Bouchiat, and D. Mailly, cond-mat/0109527 (2001).
[8] F. von Oppen and E. K. Riedel, Phys. Rev. Lett. **66**, 84 (1991).
[9] B. L. Altshuler, Y. Gefen, and Y. Imry, Phys. Rev. Lett. **66**, 88 (1991).
[10] V. Ambegaokar and U. Eckern, Phys. Rev. Lett. **65**, 381 (1990).
[11] A. Schmid, Phys. Rev. Lett. **66**, 80 (1991).
[12] L. G. Aslamazov and A. I. Larkin, Sov. Phys. JETP **40**, 321 (1975).
[13] V. Ambegaokar and U. Eckern, Europhys. Lett. **13**, 733 (1990).
[14] P. Morel and P. W. Anderson, Phys. Rev. **125**, 1263 (1962).
[15] F. B. Muller-Allinger and A. C. Mota, Phys. Rev. Lett. **84**, 3161 (2000).
[16] M. Schechter, Y. Imry, Y. Levinson, and J. von Delft, Phys. Rev. B **63**, 214518 (2001).
[17] R. W. Richardson, Phys. Lett. **3**, 277 (1963).
[18] R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964).
[19] M. Schechter, J. von Delft, Y. Imry, and Y. Levinson, Submitted to Phys. Rev. B [cond-mat/0210287] (2002).
[20] K. A. Matveev and A. I. Larkin, Phys. Rev. Lett. **78**, 3749 (1997).
[21] P. Mohanty, Ann. Phys. **8**, 549 (1999).
[22] V. E. Kravtsov and B. L. Altshuler, Phys. Rev. Lett. **84**, 3394 (2000).
[23] P. Schwab, Eur. Phys. J. B **18**, 189 (2000).
[24] Under this assumption of thin ring the response to constant magnetic field can be found along the same lines as done here for AB flux.
[25] L. D. Landau and E. M. Lifshits, *Quantum Mechanics*, 3rd ed. (Pergamon Press, Oxford, 1991).
[26] A. D. Mirlin, Phys. Rep. **326**, 259 (2000).
[27] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshindki, *Methods of quantum field theory in statistical physics* (Prentice-Hall, Englewood Cliffs, NJ, 1963).
[28] M. Schechter, Y. Oreg, Y. Imry, and Y. Levinson, , in preparation.
[29] A. D. Lorenzo, R. Fazio, F. W. J. Hekking, G. Falci, A. Mastellone, and G. Giaquinta, Phys. Rev. Lett. **84**, 550 (2000).
[30] M. Schechter, Y. Imry, Y. Levinson, and Y. Oreg, , to be published in the proceedings of Mesoscopic Superconductivity and Spintronics, Atsugi, Japan, 2002.